

Boltzmann equation

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Based on Kolb & Turner section 5

Introduction

- Macroscopically thermal equilibrium is realized when

$$\text{(Reaction rate)} \quad \Gamma > H \quad \text{(Expansion rate)} \quad (1)$$

- Treating decoupling properly you need microscopic description
Boltzmann equation

$$\hat{L}[f] = C[f] \quad (2)$$

Kinetic Theory Overview

- ▶ Particle's phase space distribution

$$f = f(p^\mu, x^\mu) \quad (3)$$

- ▶ If there is "*no collision*," f is constant along the 1-particle phase space flow

$$\frac{df}{d\tau} = 0 \quad (4)$$

- ▶ Short-range interactions (EM, nuclear int.) are introduced as *collisions* $C[f]$

Liouville's theorem

- Non-relativistic case ($f = f(\mathbf{p}, \mathbf{x}, t)$)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \quad (\text{Vlasov eq.}) \quad (5)$$

If you have non-negligible collisions, we have

$$\frac{df}{dt} = C[f] \neq 0 \quad (C[f]: \text{the collision operator}) \quad (6)$$

$$\Rightarrow \underbrace{\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{p}} \right)}_{\hat{L}: \text{non-relativistic Liouville operator}} f = C[f] \quad (\text{non-relativistic Boltzmann eq.}) \quad (7)$$

$\hat{L}$: non-relativistic Liouville operator

Liouville's theorem

► GR case ($f = f(p^\mu, x^\mu)$)

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tau} + \frac{\partial f}{\partial p^\mu} \frac{\partial p^\mu}{\partial \tau} = p^\mu \frac{\partial f}{\partial x^\mu} + \left(-\Gamma_{\alpha\beta}^\mu p^\alpha p^\beta \right) \frac{\partial f}{\partial p^\mu} \quad (8)$$

$$= C[f] \quad \left(\because \frac{\partial p^\mu}{\partial \tau} + \Gamma_{\alpha\beta}^\mu p^\alpha p^\beta = 0: \text{geodesic eq.} \right) \quad (9)$$

In FLRW metric, f is spatially homogeneous and isotropic ($\nabla f = 0, \partial f / \partial \mathbf{p} = 0$)

$$f(p^\mu, x^\mu) \rightarrow f(E(|\mathbf{p}|), t) \quad (10)$$

$$\Rightarrow \underbrace{\left(E \frac{\partial}{\partial t} - \frac{\dot{a}}{a} |\mathbf{p}|^2 \frac{\partial}{\partial E} \right)}_{\hat{L}} f = C[f] \quad (\text{FLRW Boltzmann eq.}) \quad (11)$$

$\hat{L}$: FLRW Liouville operator

Spatial Number density

- FLRW Boltzmann eq. ($C[f]$ represents collisions of the particles of energy E)

$$\left(E \frac{\partial}{\partial t} - \frac{\dot{a}}{a} |\mathbf{p}|^2 \frac{\partial}{\partial E} \right) f = C[f] \quad (12)$$

When you introduce the spatial number density (integrate over particle's energy E)

$$n(t) = g \int \frac{d^3 \mathbf{p}}{(2\pi)^3} f(E, t) \Rightarrow \frac{dn}{dt} + 3 \frac{\dot{a}}{a} n(t) = 2g \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E} C[f] = 2 \int d\Pi C[f] \quad (13)$$

$$\left(d\Pi \equiv g \frac{d^3 \mathbf{p}}{(2\pi)^3 2E} = g \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \right)$$

Collision term

► Consider a process $\underbrace{\chi + a + \cdots + d}_n \leftrightarrow \underbrace{i + \cdots + l}_m$

$$\begin{aligned}
 & 2g_\chi \int \frac{d^3 \mathbf{p}_\chi}{(2\pi)^3 2E_\chi} C[f_\chi] \\
 &= - \int d\Pi_\chi \int d\Pi_a \cdots \int d\Pi_d \int d\Pi_i \cdots \int d\Pi_l (2\pi)^4 \delta(p_\chi + p_a + \cdots + p_d - p_i - \cdots - p_l) \\
 & \quad \times \left[|\mathcal{M}_{\rightarrow}|^2 f_\chi f_a \cdots f_d (1 \pm f_i) \cdots (1 \pm f_l) - |\mathcal{M}_{\leftarrow}|^2 f_i \cdots f_l (1 \pm f_\chi) (1 \pm f_a) \cdots (1 \pm f_d) \right] \\
 & \hspace{25em} (14)
 \end{aligned}$$

+/- for bosons/fermions: $a_s^\dagger |n_s\rangle = \sqrt{1 \pm n_s} |n_s + 1\rangle$, $a_s |n_s\rangle = \sqrt{n_s} |n_s - 1\rangle$

Kinetic equilibrium

- Suppose kinetic equilibrium, then Bosons/Fermions satisfy

$$f_s(E_s) = \frac{1}{e^{\beta(E_s - \mu_s)} \mp 1} \Rightarrow 1 \pm f_s = f_s e^{\beta(E_s - \mu_s)} \quad (15)$$

Therefore, the Boltzmann equation under T symmetry (*i.e.* $|\mathcal{M}_{\leftarrow}| = |\mathcal{M}_{\rightarrow}|$) reads

$$\begin{aligned} \frac{dn_\chi}{dt} + 3\frac{\dot{a}}{a}n_\chi(t) \\ = - \int d\Pi_\chi \int d\Pi_a \cdots \int d\Pi_d \int d\Pi_i \cdots \int d\Pi_l (2\pi)^4 \delta(p_\chi + p_a + \cdots + p_d - p_i - \cdots - p_l) \\ \times |\mathcal{M}|^2 f_\chi f_a \cdots f_d f_i \cdots f_l \left[e^{\beta(E_i - \mu_i)} \cdots e^{\beta(E_l - \mu_l)} - e^{\beta(E_\chi - \mu_\chi)} e^{\beta(E_a - \mu_a)} \cdots e^{\beta(E_d - \mu_d)} \right] \end{aligned} \quad (16)$$

(RHS) vanishes when $\mu_i + \cdots + \mu_l = \mu_\chi + \mu_a + \cdots + \mu_d$.

Dilute gas limit

- Suppose the density is dilute enough $f_i \ll 1$

Then, quantum effect can be approximated by $1 \pm f_i \simeq 1$ (classical statistics apply!)

$$f_s(E_s) = \frac{1}{e^{\beta(E_s - \mu_s)} \mp 1} \rightarrow e^{-\beta(E_s - \mu_s)} \quad (17)$$

Therefore, the Boltzmann equation under T symmetry (*i.e.* $|\mathcal{M}_{\leftarrow}| = |\mathcal{M}_{\rightarrow}|$) reads

$$\begin{aligned} \frac{dn_\chi}{dt} + 3\frac{\dot{a}}{a}n_\chi(t) \\ = - \int d\Pi_\chi \int d\Pi_a \cdots \int d\Pi_d \int d\Pi_i \cdots \int d\Pi_l (2\pi)^4 \delta(p_\chi + p_a + \cdots + p_d - p_i - \cdots - p_l) \\ \times |\mathcal{M}|^2 \left[(f_\chi f_a \cdots f_d) - (f_i \cdots f_l) \right] \end{aligned} \quad (18)$$

$\chi\bar{\chi} \leftrightarrow X\bar{X}$ **weak interaction**

► Suppose $\chi\bar{\chi}$ weakly annihilate into $X\bar{X}$, which have stronger interaction
Then, $X, \bar{X}$ have thermal equilibrium distribution

$$f_X^{\text{eq}} = f_{\bar{X}}^{\text{eq}} = e^{-\beta E_X} \quad (19)$$

The Boltzmann equation for χ under T symmetry (i.e. $|\mathcal{M}_{\leftarrow}| = |\mathcal{M}_{\rightarrow}|$) reads

$$\begin{aligned} \frac{dn_\chi}{dt} + 3\frac{\dot{a}}{a}n_\chi(t) \\ = - \int d\Pi_\chi \int d\Pi_{\bar{\chi}} \int d\Pi_X \int d\Pi_{\bar{X}} (2\pi)^4 \delta(p_\chi + p_{\bar{\chi}} - p_X - p_{\bar{X}}) \\ \times |\mathcal{M}|^2 \left[f_\chi f_{\bar{\chi}} (1 \pm f_X^{\text{eq}}) (1 \pm f_{\bar{X}}^{\text{eq}}) - f_X^{\text{eq}} f_{\bar{X}}^{\text{eq}} (1 \pm f_\chi) (1 \pm f_{\bar{\chi}}) \right] \end{aligned} \quad (20)$$

$\chi\bar{\chi} \leftrightarrow X\bar{X}$ weak interaction

- The energy conservation requires

$$f_X^{\text{eq}} f_{\bar{X}}^{\text{eq}} = e^{-\beta(E_X + E_{\bar{X}})} = e^{-\beta(E_\chi + E_{\bar{\chi}})} = f_\chi^{\text{eq}} f_{\bar{\chi}}^{\text{eq}} \quad (21)$$

The Boltzmann equation for χ under the classical limit

$$\begin{aligned} \frac{dn_\chi}{dt} + 3\frac{\dot{a}}{a}n_\chi(t) &= - \int d\Pi_\chi \int d\Pi_{\bar{\chi}} \int d\Pi_X \int d\Pi_{\bar{X}} (2\pi)^4 \delta(p_\chi + p_{\bar{\chi}} - p_X - p_{\bar{X}}) |\mathcal{M}|^2 \left[f_\chi f_{\bar{\chi}} - f_\chi^{\text{eq}} f_{\bar{\chi}}^{\text{eq}} \right] \\ &= - \langle \sigma_{\text{ann}} |v| \rangle \left[n_\chi^2 - (n_\chi^{\text{eq}})^2 \right] \quad (\langle \cdot \rangle : \text{thermally-averaged quantity}) \quad (22) \\ \left(\langle \sigma_{\text{ann}} |v| \rangle \equiv \frac{1}{(n_\chi^{\text{eq}})^2} \int d\Pi_\chi \int d\Pi_{\bar{\chi}} \int d\Pi_X \int d\Pi_{\bar{X}} (2\pi)^4 \delta(p_\chi + p_{\bar{\chi}} - p_X - p_{\bar{X}}) |\mathcal{M}|^2 f_\chi^{\text{eq}} f_{\bar{\chi}}^{\text{eq}} \right) \end{aligned}$$

Entropy conservation

Boltzmann equation

$$\frac{dn_\chi}{dt} + 3Hn_\chi(t) = -\langle\sigma_{\text{ann}}|v|\rangle \left[n_\chi^2 - (n_\chi^{\text{eq}})^2 \right] \quad (23)$$

Cosmic entropy is conserved during the adiabatic expansion, so let us define

$$Y_\chi = \frac{n_\chi}{s} \text{ (yield)}, \quad x_\chi = \frac{m_\chi}{T}, \quad (24)$$

where Y_χ represents the *actual* number of χ per comoving volume. Then

$$\frac{dn_\chi}{dt} + 3Hn_\chi(t) = \frac{1}{a^3} \frac{d}{dt} (n_\chi a^3) = \frac{1}{a^3} \frac{d}{dt} (Y_\chi (sa^3)) = s \frac{dY_\chi}{dt} \quad (25)$$

Since in RD epoch, $T \propto a^{-1} \propto t^{-1/2}$, then $H = 1/2t$, $x \propto m_\chi t^{1/2}$. This implies

$$\frac{dt}{dx} = \frac{1}{xH} \quad (26)$$

Final form

Boltzmann equation

$$\frac{dn_\chi}{dt} + 3Hn_\chi(t) = -\langle\sigma_{\text{ann}}|v|\rangle s^2 \left[Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right] \quad (27)$$

In RD epoch, we have

$$(\text{LHS}) = s \frac{dY_\chi}{dt}, \quad \frac{dt}{dx} = \frac{1}{xH} \quad (28)$$

Then we get

$$\frac{dY_\chi}{dx} = -\frac{\langle\sigma_{\text{ann}}|v|\rangle s(T)}{xH(T)} \left[Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right] \quad (29)$$

You can solve this numerically.

Numerical simulation

Boltzmann equation

$$\frac{dY_\chi}{dx} = -\frac{\langle \sigma_{\text{ann}} |v| \rangle s(T)}{xH(T)} \left[Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right] \quad (30)$$

In numerical simulation, it is often introduced temperature independent variables

$$s(T) =: \frac{s(m)}{x^3}, \quad H(T) =: \frac{H(m)}{x^2} \quad (31)$$

Then we get

$$\frac{dY_\chi}{dx} = -\frac{\langle \sigma_{\text{ann}} |v| \rangle s(m)}{x^2 H(m)} \left[Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right] \quad (32)$$

$$\Rightarrow \frac{dy_\chi}{dx} = -\frac{\langle \sigma_{\text{ann}} |v| \rangle}{x^2} \left[y_\chi^2 - (y_\chi^{\text{eq}})^2 \right] \quad \left(y_\chi = \frac{s(m)}{H(m)} Y_\chi \right) \quad (33)$$

Conclusion

- Boltzmann equation for weakly interacting dilute system

$$\frac{dY_\chi}{dx} = -\frac{\langle\sigma_{\text{ann}}|v|\rangle s(T)}{xH(T)} \left[Y_\chi^2 - (Y_\chi^{\text{eq}})^2 \right] \quad (34)$$

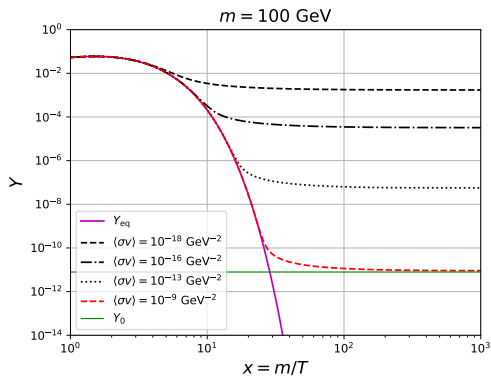


Figure: Freeze-out simulations for S-wave case: $\langle\sigma_{\text{ann}}|v|\rangle = \text{const.}$

Comparison between S-wave & P-wave

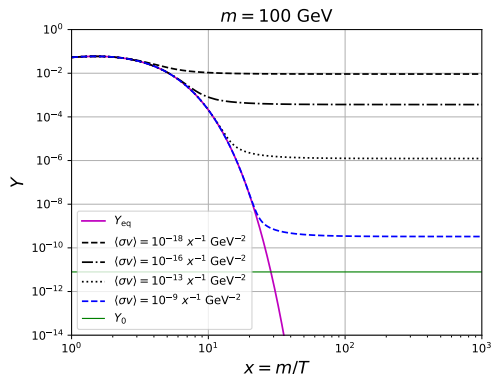


Figure: P-wave case: $\langle \sigma_{\text{ann}} |v| \rangle \propto x^{-1}$

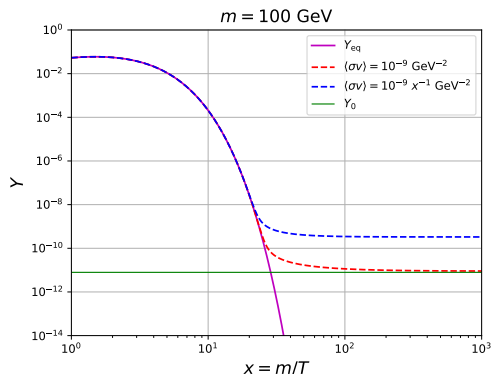


Figure: S-wave vs P-wave